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Coherence Spaces and Geometry of Interaction

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1 Introduction

There are two styles in the study of denotational semantics for functional programming languages: modeling programs as functions and modeling programs as strategies. While the former is good at capturing static features of programming languages, the latter is good at capturing dynamic features of programming languages. In [9], Oosten gives a result that connects the former to the latter. He shows that the finite types in the coKleisli category of Hypercoherence spaces [3], where programs are interpreted as strongly stable functions, coincides with the finite types in the realizability model based on a combinatory algebra $\mathscr{B}$. In this realizability model, programs are interpreted as equivalence classes of strategies. As pointed out in [7], we can construct $\mathscr{B}$ following categorical Geometry of Interaction introduced by Abramsky, Haghverdi and Scott [1]. In this paper, we give a similar result for coherence spaces [4]. We relate the category of coherence spaces and stable functions with a realizability model whose construction is based on categorical Geometry of Interaction.

2 Outline

In Section 3, we recall coherence spaces and stable functions between them and we describe the cartesian closed structure of the category **Coh** of coherence spaces and stable functions. In Section 4, we construct an SK-algebra $\mathscr{R}$ following categorical Geometry of Interaction introduced by Abramsky, Haghverdi and Scott [1]. This SK-algebra consists of $\mathbb{N}$-by-$\mathbb{N}$ matrices over the semiring $\mathbb{N}\cup\{\infty\}$. We can regard $\mathbb{N}$-by-$\mathbb{N}$ matrices as representations of execution strategies of programs. In Section 5, we give a cartesian closed category $\mathbf{Asm}(\mathscr{R})$ consisting of sets and functions "realized by $\mathscr{R}$", and in Section 6, we show that there is a full cartesian closed subcategory of **Coh** that is equivalent to a full subcategory of $\mathbf{Asm}(\mathscr{R})$.

3 Coherent Space

Coherence spaces are introduced by Girard [4]. We recall the definition of coherent spaces and stable functions between them and give some basic fact about coherence spaces.

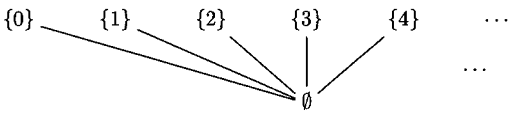
Definition 3.1. *A* coherent space $\mathcal{X}$ *is a pair* $(|\mathcal{X}|, \frown\!\!\!\!\smile_{\mathcal{X}})$ *of a set* $|\mathcal{X}|$ *and a reflexive binary relation* $\frown\!\!\!\!\smile_{\mathcal{X}}$ *on* $|\mathcal{X}|$. *A* clique *of a coherent space* $\mathcal{X}$ *is a subset* $a \subseteq |\mathcal{X}|$ *such that* $x \frown\!\!\!\!\smile_{\mathcal{X}} y$ *for all* $x, y \in a$. *We write* $x \frown_{\mathcal{X}} y$ *when* $x \frown\!\!\!\!\smile_{\mathcal{X}} y$ *and* $x \neq y$.

We write $\mathcal{C}(\mathcal{X})$ for the set of cliques of $\mathcal{X}$ and $\mathcal{C}_{\mathrm{fin}}(\mathcal{X})$ for the set of finite cliques. With respect to the inclusion order, $\mathcal{C}(\mathcal{X})$ forms a pointed directed complete poset (dcpo). We always regard $\mathcal{C}(\mathcal{X})$ as a dcpo in this way. For some technical reasons, we only consider coherence spaces whose underlying set is at most countable.

Example 3.1. *We define a coherence space $\mathcal{N}$ by*

$$|\mathcal{N}| = \mathbb{N} = \{0, 1, 2, \ldots\},$$
$$n \asymp_{\mathcal{N}} m \iff n = m$$

The dcpo $\mathcal{C}(\mathcal{N})$ is the following flat domain:



Definition 3.2. *Let $\mathcal{X}$ and $\mathcal{Y}$ be coherent spaces. A* stable *function $f\colon \mathcal{C}(\mathcal{X}) \to \mathcal{C}(\mathcal{Y})$ is a continuous function that preserves meets of bounded cliques:*

$$u \cup v \in \mathcal{C}(\mathcal{X}) \implies f(u \cap v) = f(u) \cap f(v)$$

for all $u, v \in \mathcal{C}(\mathcal{X})$.

We define **Coh** to be the category of coherent spaces and stable functions.

Theorem 3.1. *The category* **Coh** *is a cartesian closed category.*

We describe the cartesian closed structure of **Coh**. The terminal object $\mathcal{T}$ is given by $|\mathcal{T}| = \emptyset$. The product of coherence spaces $\mathcal{X}$ and $\mathcal{Y}$ is given by

$$|\mathcal{X} \times \mathcal{Y}| = |\mathcal{X}| + |\mathcal{Y}| = \{(0, x) \mid x \in |\mathcal{X}|\} \cup \{(1, y) \mid y \in |\mathcal{Y}|\}$$

and

$$(i, z) \asymp_{\mathcal{X} \times \mathcal{Y}} (j, w) \iff (i = j = 0 \text{ and } z \asymp_{\mathcal{X}} w) \text{ or } (i = j = 1 \text{ and } z \asymp_{\mathcal{Y}} w).$$

For coherent spaces $\mathcal{X}$ and $\mathcal{Y}$, the underlying set of the exponential $\mathcal{X} \Rightarrow \mathcal{Y}$ is given by

$$|\mathcal{X} \Rightarrow \mathcal{Y}| = \mathcal{C}_{\mathrm{fin}}(\mathcal{X}) \times |\mathcal{Y}|$$

and

$$(u, y) \asymp_{\mathcal{X} \Rightarrow \mathcal{Y}} (u', y') \iff (u \cup u' \in \mathcal{C}(\mathcal{X}) \implies y \asymp_{\mathcal{Y}} y') \land (u \neq u' \land u \cup u' \in \mathcal{C}(\mathcal{X}) \implies y \frown_{\mathcal{Y}} y')\ .$$

For more detail, see [5].

As is known, there is a bijective correspondence between the set of stable functions from $\mathcal{X}$ to $\mathcal{Y}$ and the set of cliques of $\mathcal{X} \Rightarrow \mathcal{Y}$. A stable function $f\colon \mathcal{C}(\mathcal{X}) \to \mathcal{C}(\mathcal{Y})$ gives rise to a clique $\mathrm{Trace}(f)$ given by

$$\mathrm{Trace}(f) = \{(u, y) \in \mathcal{C}_{\mathrm{fin}}(X) \times |Y| \mid u \text{ is the least clique such that } y \in f(u)\},$$

and a clique $u \in \mathcal{C}(X \Rightarrow Y)$ gives rise to a stable function $\mathrm{Fun}(u)$ given by

$$(\mathrm{Fun}(u))\,(v) = \{y \in |Y| \mid (v', y) \in u \text{ for some } v' \subseteq v\}.$$

We call $\mathrm{Trace}(f)$ the *trace of f*.

We give a stable function and a non-stable function.

Example 3.2. *We define $\varphi_{\mathsf{pconv}}\colon \mathcal{C}(\mathcal{N} \times \mathcal{N}) \to \mathcal{C}(\mathcal{N})$ by*

$$\varphi_{\mathsf{pconv}}(a) = \begin{cases} \{0\}, & \text{if } a \text{ is not empty,} \\ \emptyset, & \text{if } a \text{ is empty.} \end{cases}$$

This function is continuous. However, φ_{pconv} is not stable since

$$\varphi_{\mathsf{pconv}}(\{(0,0)\}) \cap \varphi_{\mathsf{pconv}}(\{(1,0)\}) \neq \varphi_{\mathsf{pconv}}(\emptyset).$$

Example 3.3. *A function* $\varphi_{\mathsf{gustave}}\colon \mathcal{C}(\mathcal{N} \times (\mathcal{N} \times \mathcal{N})) \to \mathcal{C}(\mathcal{N})$ *given by*

$$\varphi_{\mathsf{gustave}}(a) = \begin{cases} \{0\}, & \textit{if } \{(0,0),(1,(0,1))\} \subseteq a, \\ \{0\}, & \textit{if } \{(1,(0,0)),(1,(1,1))\} \subseteq a, \\ \{0\}, & \textit{if } \{(1,(1,0)),(0,1)\} \subseteq a, \\ \emptyset, & \textit{otherwise} \end{cases}$$

is stable.

4 Geometry of Interaction

4.1 Weighted relation

We naturally extend arithmetic operations on $\mathbb{N}$ to operations on $\mathbb{N} \cup \{\infty\}$:

$$x + \infty = \infty + x = \infty;$$

$$x \cdot \infty = \infty \cdot x = \begin{cases} \infty, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

We note that $(\mathbb{N} \cup \{\infty\}, +, \cdot, 0, 1)$ is a commutative semiring. We extend this binary summation to countable summation:

$$\sum_{i \in I} x_i = \begin{cases} \sum_{i \in I_0} x_i, & \text{if there is a finite } I_0 \subseteq I \text{ such that } x_i = 0 \text{ for all } i \in I \setminus I_0 \\ \infty, & \text{otherwise.} \end{cases}$$

We define a category **WRel** by:

- objects are sets;
- morphisms from X to Y are functions from $X \times Y$ to $\mathbb{N} \cup \{\infty\}$.

The identity $\mathrm{id}_X \colon X \to X$ is given by

$$\mathrm{id}_X(x, x') = \begin{cases} 1, & \text{if } x = x', \\ 0, & \text{if } x \neq x', \end{cases}$$

and composition of $f \colon X \to Y$ and $g \colon Y \to Z$ is given by

$$(g \circ f)(x, z) = \sum_{y \in Y} g(y, z) f(x, y).$$

We can regard morphisms in $\mathbf{WRel}(X, Y)$ as weighted relations (or directed graphs) between X and Y.

WRel has countable biproducts. The zero object is the emptyset $\emptyset$. The biproduct of $X_0, X_1, X_2, \ldots \in$ **WRel** is the disjoint sum

$$\bigoplus_{n \in \mathbb{N}} X_n = \bigcup_{n \in \mathbb{N}} \{(n, x) \mid x \in X_n\}$$

with injections $\iota_i \colon X_i \to \bigoplus_{n \in \mathbb{N}} X_n$ given by

$$\iota_i(x, (n, x')) = \begin{cases} 1, & \text{if } n = i \text{ and } x = x', \\ 0, & \text{otherwise,} \end{cases}$$

and projections $\pi_i \colon X_i \to \bigoplus_{n\in\mathbb{N}} X_n$ given by

$$\pi_i((n,x),x') = \begin{cases} 1, & \text{if } n = i \text{ and } x = x', \\ 0, & \text{otherwise.} \end{cases}$$

For $\{f_n \colon X_n \to Y_n\}_{n\in\mathbb{N}}$, the biproduct $\bigoplus_{n\in\mathbb{N}} f_n \colon \bigoplus_{n\in\mathbb{N}} X_n \to \bigoplus_{n\in\mathbb{N}} Y_n$ is given by

$$\left(\bigoplus_{n\in\mathbb{N}} f_n\right)((i,x),(j,y)) = \begin{cases} f_i(x,y), & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

The countable biproducts induce countable summation on hom-sets $\mathbf{WRel}(X,Y)$: for a countable family $\{f_i \colon X \to Y\}_{i\in I}$, we define $\sum_{i\in I} f_i \colon X \to Y$ by

$$\left(\sum_{i\in I} f_i\right)(x,y) = \sum_{i\in I} f_i(x,y).$$

As a corollary of [8, Theorem 3], we obtain a uniform trace operator on **WRel**.

Proposition 4.1. $(\mathbf{WRel}, \emptyset, \oplus)$ *has a uniform trace operator* tr *given by*

$$\mathrm{tr}^Z_{X,Y}(f \colon X \oplus Z \to Y \oplus Z) = f_{00} + \sum_{n\in\mathbb{N}} f_{01} \circ f_{11}^n \circ f_{10}$$

where

$$\begin{aligned} f_{00} &\colon X \to Y \\ f_{01} &\colon X \to Z \\ f_{10} &\colon Z \to Y \\ f_{11} &\colon Z \to Z \end{aligned}$$

are the restrictions of f.

4.2 SK-algebra

Following [1], we construct an SK-algebra from a *GoI situation* ([1, Definition 4.1]) on **WRel**. We first define a traced strong monoidal functor $F \colon \mathbf{WRel} \to \mathbf{WRel}$ by

$$FX = \mathbb{N} \times X$$

and

$$(F(f \colon X \to Y))((n,x),(m,y)) = \begin{cases} f(x,y), & \text{if } n = m, \\ 0, & \text{otherwise.} \end{cases}$$

We define isomorphisms

$$\alpha \colon \mathbb{N} \oplus \mathbb{N} \to \mathbb{N}, \qquad \beta \colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

by

$$\alpha((i,n),m) = \begin{cases} 1, & i = 0 \text{ and } 2n = m, \\ 1, & i = 1 \text{ and } 2n + 1 = m, \\ 0, & \text{otherwise,} \end{cases}$$

$$\beta((n,m),k) = \begin{cases} 1, & \text{if } \dfrac{(n+m)(n+m+1)}{2} + m = k, \\ 0, & \text{otherwise.} \end{cases}$$

In the sequel, we write $\langle n, m\rangle$ for $\dfrac{(n+m)(n+m+1)}{2} + m$. We note that $\langle -, -\rangle$ is a bijection from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$.

We define pointwise monoidal natural transformations

$$\begin{aligned} d_X &: FX \to X, \\ \delta_X &: FX \to FFX, \\ w_X &: FX \to \emptyset, \\ c_X &: FX \to FX \oplus FX. \end{aligned}$$

by

$$\begin{aligned} d_X((n,x),y) &= \begin{cases} 1, & \text{if } n = 0 \text{ and } x = y, \\ 0, & \text{otherwise,} \end{cases} \\ \delta_X((n,x),(m,(k,y))) &= \begin{cases} 1, & \text{if } \langle m,k\rangle = n \text{ and } x = y, \\ 0, & \text{otherwise,} \end{cases} \\ w_X &= \text{the unique morphism from } FX \text{ to } \emptyset, \\ c_X((n,x),(i,(m,y))) &= \begin{cases} 1, & \text{if } i + 2m = n \text{ and } x = y, \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

and we define pointwise monoidal natural transformations

$$\begin{aligned} d'_X &: X \to FX, \\ \delta'_X &: FFX \to FX, \\ w'_X &: \emptyset \to FX, \\ c'_X &: FX \oplus FX \to FX. \end{aligned}$$

by

$$\begin{aligned} d'_X(y,(n,x)) &= \begin{cases} 1, & \text{if } n = 0 \text{ and } x = y, \\ 0, & \text{otherwise,} \end{cases} \\ \delta'_X((m,(k,y)),(n,x)) &= \begin{cases} 1, & \text{if } \langle m,k\rangle = n \text{ and } x = y, \\ 0, & \text{otherwise,} \end{cases} \\ w'_X &= \text{the unique morphism from } \emptyset \text{ to } FX, \\ c'_X((i,(m,y)),(n,x)) &= \begin{cases} 1, & \text{if } i + 2m = n \text{ and } x = y, \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

Proposition 4.2. $(\mathbf{WRel}, \mathbb{N}, F, \alpha, \alpha^{-1}\beta, \beta^{-1}, d, d', \delta, \delta', w, w', c, c')$ *is a* GoI situation.

Corollary 4.1. $\mathbf{WRel}(\mathbb{N}, \mathbb{N})$ *with a binary application*

$$a \cdot b = \mathrm{tr}^{\mathbb{N}}_{\mathbb{N},\mathbb{N}}(\alpha^{-1} \circ a \circ \alpha \circ (\mathrm{id}_{\mathbb{N}} \oplus (\beta \circ Fb \circ \beta^{-1})))$$

is an SK-algebra. As a consequence, for any formal expression $e(x)$ generated by a variable x, elements in $\mathbf{WRel}(\mathbb{N}, \mathbb{N})$ *and the binary application, there is $\lambda x.\, e(x) \in \mathbf{WRel}(\mathbb{N},\mathbb{N})$ such that for any $a \in \mathbf{WRel}(\mathbb{N},\mathbb{N})$, $(\lambda x.\, e(x)) \cdot a = e(a)$.*

In the sequel, we denote this SK-algebra by $\mathscr{R}$. Let $\leq$ be the pointwise order on $\mathscr{R}$:

$$a \leq b \iff a(n,m) \leq b(n,m) \text{ for all } (n,m) \in \mathbb{N} \times \mathbb{N}.$$

It is easy to check that $\mathscr{R}$ forms a complete lattice with respect to this partial order. The least element $\varnothing \in \mathscr{R}$ is given by

$$\varnothing(n,m) = 0,$$

and the least upper bound of $\{a_i \in \mathscr{R}\}_{i \in I}$ is given by

$$\left(\bigvee_{i \in I} a_i\right)(n,m) = \bigvee_{i \in I} a_i(n,m).$$

By unfolding the definition of the binary application of $\mathscr{R}$, we can check that $(-)\cdot(-)$ preserves joins.

Proposition 4.3. *For all $a \in \mathscr{R}$,*

$$(-)\cdot a,\ a \cdot (-) \colon \mathscr{R} \to \mathscr{R}$$

are join-preserving functions.

Lemma 4.1. *For any family $\{r_n \in \mathscr{R}\}_{n \in \mathbb{N}}$, there is $s \in \mathscr{R}$ such that*

$$s \cdot x = \sum_{n \in \mathbb{N}} r_n \cdot x.$$

Proof. $s \in \mathscr{R}$ given by

$$\begin{aligned} s(2n, 2m) &= \sum_{i \in \mathbb{N}} r_i(2n, 2m), \\ s(2n, 2\langle\langle i, m_0\rangle, m_1\rangle + 1) &= r_i(2n, 2\langle m_0, m_1\rangle + 1), \\ s(2\langle\langle i, n_0\rangle, n_1\rangle + 1, 2m) &= r_i(2\langle n_0, n_1\rangle + 1, 2m), \\ s(2\langle\langle i, n_0\rangle, n_1\rangle + 1, 2\langle\langle i, m_0\rangle, m_1\rangle + 1) &= r_i(2\langle n_0, n_1\rangle + 1, 2\langle m_0, m_1\rangle + 1) \end{aligned}$$

satisfies $s \cdot x = \sum_{n \in \mathbb{N}} r_n \cdot x$. ☐

We define $\mathsf{I} \in \mathscr{R}$ by

$$\mathsf{I}(n,m) = \begin{cases} 1, & \text{if } n \text{ is even and } m = 2\left\langle 0, \dfrac{n}{2}\right\rangle + 1, \\ 1, & \text{if } m \text{ is even and } n = 2\left\langle 0, \dfrac{m}{2}\right\rangle + 1, \\ 0, & \text{otherwise.} \end{cases}$$

For any $a \in \mathscr{R}$, we have $\mathsf{I} \cdot a = a$.

Lemma 4.2. *Let r, s, s', s'' be elements of $\mathscr{R}$ such that $s \leq s''$ and $s' \leq s''$. If $r \cdot s = \mathsf{I}$ and $r \cdot s' = \mathsf{I}$ and $r \cdot s'' = \mathsf{I}$, then $r \cdot (s \wedge s') = \mathsf{I}$.*

Proof. By monotonicity of $r \cdot (-)$,

$$r \cdot (s \wedge s') \leq r \cdot s = \mathsf{I}.$$

For any $n, m \in \mathbb{N}$ such that $\mathsf{I}(n,m) = 1$, since $r \cdot s' = \mathsf{I}$ and $r \cdot s'' = \mathsf{I}$, there are finite sequences $(k_1, \ldots, k_i), (k_1, \ldots, k'_{2i})$ such that

$$\begin{aligned} r(2n, 2\langle k_1, k'_1\rangle + 1) &= 1 \\ s'(k'_1, k'_2) &= 1 \\ r(2\langle k_1, k'_2\rangle + 1, 2\langle k_2, k'_3\rangle + 1) &= 1 \\ s'(k'_3, k'_4) &= 1 \\ &\vdots \\ r(2\langle k_{i-1}, k'_{2i-2}\rangle + 1, 2\langle k_i, k'_{2i-1}\rangle + 1) &= 1 \\ s'(k'_{2i-1}, k'_{2i}) &= 1 \\ r(2\langle k_i, k'_{2i}\rangle + 1, 2m) &= 1 \end{aligned}$$

and $(l_1, \ldots, l_j), (l_1, \ldots, l'_{2j})$ such that

$$\begin{aligned}
r(2n, 2\langle l_1, l'_1\rangle + 1) &= 1\\
s''(l'_1, l'_2) &= 1\\
r(2\langle l_1, l'_2\rangle + 1, 2\langle l_2, l'_3\rangle + 1) &= 1\\
s''(l'_3, l'_4) &= 1\\
&\vdots\\
r(2\langle l_{j-1}, l'_{2j-2}\rangle + 1, 2\langle l_j, l'_{2j-1}\rangle + 1) &= 1\\
s''(l'_{2j-1}, l'_{2j}) &= 1\\
r(2\langle l_j, l'_{2j}\rangle + 1, 2m) &= 1.
\end{aligned}$$

If $((k_1, \cdots, k_i), (k_1, \cdots, k'_{2i})) \neq ((l_1, \ldots, l_j), (l_1, \ldots, l'_{2j}))$, then $\mathsf{I} = r \cdot s \geq 2\mathsf{I}$. Hence, $i = j$ and

$$((k_1, \cdots, k_j), (k_1, \cdots, k'_{2j})) = ((l_1, \ldots, l_j), (l_1, \ldots, l'_{2j})).$$

In particular, $r \cdot (s \wedge s') \geq \mathsf{I}$. □

5 The Category of Assemblies

Definition 5.1. *An* assembly X *on* $\mathscr{R}$ *is a set* $|X|$ *equipped with a function*

$$\|-\|_X \colon |X| \to \mathcal{P}^+(\mathscr{R})$$

where $\mathcal{P}^+(\mathscr{R})$ *is the set of all nonempty subsets of* $\mathscr{R}$.

Intuitively, the underlying set $|X|$ of an assembly X is the set of "values" and $\|x\|_X \subseteq R$ are "implementations" of x.

Example 5.1. *We define* N *by*

$$|N| = \{\bot\} \cup \mathbb{N}\,, \qquad \|x\| = \begin{cases} \{\varnothing\}\,, & \text{if } x \text{ is } \bot,\\ \{c_x\}\,, & \text{if } x \text{ is a natural number}\end{cases}$$

where we define $c_n \in \mathscr{R}$ *by*

$$c_n(i,j) = \begin{cases} 1\,, & \text{if } i = 0 \text{ and } j = n,\\ 0\,, & \text{otherwise.}\end{cases}$$

Definition 5.2. *For assemblies* X *and* Y*, a function* f *from* $|X|$ *to* $|Y|$ *is* realizable *when there is* $r \in \mathscr{R}$ *such that for any* $x \in X$*, if* $a \in \|x\|_X$*, then* $r \cdot a \in \|fx\|_Y$*. We call* r *a* realizer *of* f.

We write $\mathbf{Asm}(\mathscr{R})$ for the category of assemblies on $\mathscr{R}$ and realizable functions between them.

Proposition 5.1. *The category* $\mathbf{Asm}(\mathscr{R})$ *is a cartesian closed category.*

Proof. See [6]. □

We describe the cartesian closed structure of $\mathbf{Asm}(\mathscr{R})$. The terminal object $T \in \mathbf{Asm}(\mathscr{R})$ is given by

$$\begin{aligned}|T| &= \{*\},\\ \|*\|_T &= \mathscr{R}.\end{aligned}$$

For $X, Y \in \mathbf{Asm}(\mathscr{R})$, the product $X \times Y$ is given by

$$|X \times Y| = |X| \times |Y| \, ,$$
$$\|(x,y)\|_{X\times Y} = \{r \boxplus s \mid r \in \|x\|_X \text{ and } s \in \|y\|_Y\}.$$

where $r \boxplus s = \alpha \circ (r \oplus s) \circ \alpha^{-1}$. The first projection $\pi_{X,Y} \colon X \times Y \to X$ is realized by $\mathsf{fst} \in \mathscr{R}$ given by

$$\mathsf{fst}(n,m) = \begin{cases} 1, & \text{if } n \text{ is even and } m = 2\langle 0, 2n\rangle + 1, \\ 1, & \text{if } m \text{ is even and } n = 2\langle 0, 2m\rangle + 1, \\ 0, & \text{otherwise,} \end{cases}$$

and the second projection $\pi'_{X,Y} \colon X \times Y \to Y$ is realized by $\mathsf{snd} \in \mathscr{R}$ given by

$$\mathsf{snd}(n,m) = \begin{cases} 1, & \text{if } n \text{ is even and } m = 2\langle 0, 2n+1\rangle + 1, \\ 1, & \text{if } m \text{ is even and } n = 2\langle 0, 2m+1\rangle + 1, \\ 0, & \text{otherwise.} \end{cases}$$

The exponential $X \Rightarrow Y$ is given by

$$|X \Rightarrow Y| = \{f \colon |X| \to |Y| \mid f \text{ is realizable}\} \, ,$$
$$\|f\|_{X \Rightarrow Y} = \{r \in \mathscr{R} \mid r \text{ is a realizer of } f\} \, .$$

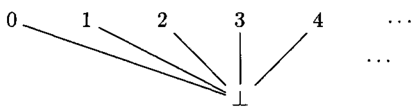
Remark 5.1. *In [6], $\|(x,y)\|_{X\times Y}$ is given by*

$$\|(x,y)\|_{X\times Y} = \{\lambda k.\, k \cdot r \cdot s \mid r \in \|x\|_X \text{ and } s \in \|y\|_Y\} \, .$$

This is essentially equivalent to our definition of $\|(x,y)\|_{X\times Y}$ because there are $p, q \in \mathscr{R}$ such that for all $a, b \in \mathscr{R}$,

$$p \cdot (a \boxplus b) = \lambda k.\, k \cdot r \cdot s,$$
$$q \cdot (\lambda k.\, k \cdot r \cdot s) = a \boxplus b.$$

We illustrate the exponential of $\mathbf{Asm}(\mathscr{R})$. The underlying set of $N \Rightarrow N$ is the set of monotone functions from $N \cup \{\perp\}$ to $N \cup \{\perp\}$ where we regard $N \cup \{\perp\}$ as the following poset



We can check this fact as follows. First, if $f \colon \{\perp\} \cup \mathbb{N} \to \{\perp\} \cup \mathbb{N}$ is realizable, then by Proposition 4.3, f must be monotone. Next, if $f \colon \{\perp\} \cup \mathbb{N} \to \{\perp\} \cup \mathbb{N}$ is a monotone function, then $r \in \mathscr{R}$ given by

$$r(n,m) = \begin{cases} 1 \, , & \text{if } n = 0 \text{ and } m = 1, \\ 1 \, , & \text{if } n = 2\langle 0, m\rangle + 1 \text{ and } m = 2f(k), \\ 0 \, , & \text{otherwise} \end{cases}$$

realizes f.

To illustrate how weights work in $\mathbf{Asm}(\mathscr{R})$, we give an example of a function that is not realizable. We define a function $p \colon |N| \times |N| \to |N|$ by

$$p(x,y) = \begin{cases} 0 \, , & \text{if } x \text{ or } y \text{ is } 0, \\ \perp \, , & \text{otherwise.} \end{cases}$$

We suppose that p is realizable and derive contradiction. If p is realizable, then there is $r \in \mathscr{R}$ such that

$$r \cdot (c_0 \boxplus c_0) = c_0\ , \qquad r \cdot (c_0 \boxplus \varnothing) = c_0\ , \qquad r \cdot (\varnothing \boxplus c_0) = c_0 \qquad r \cdot (\varnothing \boxplus \varnothing) = \varnothing$$

Then, by Lemma 4.2,

$$\varnothing = r \cdot (\varnothing \boxplus \varnothing) = (r \cdot (c_0 \boxplus \varnothing)) \wedge (r \cdot (\varnothing \boxplus c_0)) = c_0.$$

We explain the fact that p is not realizable in terms of Game semantics [2]. The condition $r \cdot (c_0 \boxplus \varnothing) = c_0$ means that r must check whether the first argument is a value or not without throwing any question to the second argument, and if the first argument is a value then r must return 0. Similarly, The condition $r \cdot (\varnothing \boxplus c_0) = c_0$ means that r must check whether the second argument is a value or not without querying about the first argument, and if the second argument is a value then r must return 0. Therefore, there must be two interaction paths in the execution of $(r \cdot (c_0 \boxplus c_0))(0,0)$:

Hence, $(r \cdot (c_n \boxplus c_m))(0,0) = c_0(0,0) = 1$ must be greater than or equal to 2.

6 Simulation between Asm($\mathscr{R}$) and Coh

Below, given a triple $(X, \mathcal{X}, \theta)$ consisting of an assembly X, a coherence space $\mathcal{X}$ and a bijection $\theta\colon |X| \to \mathcal{C}(\mathcal{X})$, we always identify X with the set of cliques $\mathcal{C}(\mathcal{X})$ via θ. For example, a bounded subset $D \subseteq |X|$ means a subset of $|X|$ that is mapped to a bounded subset of $\mathcal{C}(\mathcal{X})$ by θ. A finite element of $|X|$ means an element $x \in |X|$ such that $\theta(x) \in \mathcal{C}(\mathcal{X})$ is finite.

Let Σ be an assembly given by

$$|\Sigma| = \{0, 1\}, \qquad\qquad \|x\|_\Sigma = \begin{cases} \{\varnothing\}, & \text{if } x = 0, \\ \{\mathsf{I}\}, & \text{if } x = 1. \end{cases}$$

Definition 6.1. *A* coherent assembly *is a triple $(X, \mathcal{X}, \theta)$ consisting of an assembly X with a coherence space $\mathcal{X}$ and a bijection $\theta\colon |X| \to \mathcal{C}(\mathcal{X})$ subject to the following conditions.*

1. There is a family $\{r_u \in \|u\|_X\}_{u \in |X|}$ such that

- $r_\emptyset = \varnothing$,
- $r_{u \cap v} = r_u \wedge r_v$ *for any bounded pair $u, v \in |X|$,*
- $r_{\bigcup D} = \bigvee_{u \in D} r_u$ *for any bounded subset $D \subseteq |X|$,*

2. For any finite $u \in |X|$, a function $\sigma_u\colon |X| \to |\Sigma|$ given by

$$\sigma_u(v) = \begin{cases} 0, & \text{if } v \not\supseteq u, \\ 1, & \text{if } v \supseteq u, \end{cases}$$

is a realizable function from X to Σ.

For example, $(N, \mathcal{N}, \theta)$ is a coherent assembly where θ is given by

$$\theta(\bot) = \emptyset, \qquad \theta(n) = \{n\}.$$

For simplicity, for every coherent assembly $(X, \mathcal{X}, \theta)$, we fix $\{r_u \in \|u\|_X\}_{u \in |X|}$ and a realizer d_u of σ_u for each $u \in |X|$ that satisfy the conditions in Definition 6.1.

Lemma 6.1. *Let $(X, \mathcal{X}, \theta)$ be a coherent assembly. For all $u, v \in |X|$,*

$$u \subseteq v \iff \textit{there are } r \in \|u\|_X \textit{ and } s \in \|v\|_X \textit{ such that } r \leq s.$$

Proof. If $u \subseteq v$, then $\{u, v\}$ is directed. Hence, $r_u \leq r_v$. If there are $r \in \|u\|_X$ and $s \in \|v\|_X$ such that $r \leq s$, then for any finite $u' \in |X|$,

$$\begin{aligned} u' \subseteq u &\implies d_{u'} \cdot r = \mathsf{I} \\ &\implies d_{u'} \cdot s = \mathsf{I} \\ &\implies u' \subseteq v. \end{aligned}$$

Hence, $u \subseteq v$. □

Lemma 6.2. *For coherent assemblies $(X, \mathcal{X}, \theta)$ and $(Y, \mathcal{Y}, \xi)$, if $f \colon |X| \to |Y|$ is realizable, then f is continuous.*

Proof. Monotonicity of f follows from Lemma 6.1. We choose a realizer $p \Vdash f$. For any directed subset $D \subseteq |X|$, the lub $\bigcup D$ is mapped to $f(\bigcup D)$, which is realized by

$$p \cdot r_{\bigcup D} = p \cdot \left(\bigvee_{u \in D} r_u \right) = \bigvee_{u \in D} p \cdot r_u.$$

For any finite $v \in |X|$,

$$\begin{aligned} v \subseteq f\left(\bigcup D\right) &\iff d_v \cdot \left(\bigvee_{u \in D} p \cdot r_u \right) = \mathsf{I} \\ &\iff d_v \cdot (p \cdot r_u) = \mathsf{I} \text{ for some } u \in D \\ &\iff v \subseteq f(u) \text{ for some } u \in D \\ &\iff v \subseteq \bigcup_{u \in D} f(u). \end{aligned}$$

Hence, $f\left(\bigcup D\right) = \bigcup_{d \in D} f(d)$. □

Lemma 6.3. *For coherent assemblies $(X, \mathcal{X}, \theta)$ and $(Y, \mathcal{Y}, \xi)$, if $f \colon |X| \to |Y|$ is realizable, then f is stable.*

Proof. We choose a realizer $p \Vdash f$. Let $u, v \in |X|$ be a bounded pair. For any finite $w \in |X|$,

$$\begin{aligned} w \subseteq f(u) \cap f(v) &\implies d_w \cdot (p \cdot r_u) = \mathsf{I} \text{ and } d_w \cdot (p \cdot r_v) = \mathsf{I} \text{ and } d_w \cdot (p \cdot r_{u \cup v}) = \mathsf{I} \\ &\implies (\lambda x.\, d_w \cdot (p \cdot x)) \cdot (r_u \wedge r_v) = \mathsf{I} \\ &\implies d_w \cdot (p \cdot (r_u \wedge r_v)) = \mathsf{I} \\ &\implies d_w \cdot (p \cdot r_{u \cap v}) = \mathsf{I} \\ &\implies w \subseteq f(u \cap v). \end{aligned}$$

Hence, $f(u) \cap f(v) = f(u \cap v)$. □

Lemma 6.4. *For coherent assemblies $(X, \mathcal{X}, \theta)$ and $(Y, \mathcal{Y}, \xi)$, if $f \colon |X| \to |Y|$ is a stable function, then f is realizable.*

Proof. Let T be the trace of f, i.e.,

$$T = \{(u, y) \in \mathcal{C}_{\text{fin}}(\mathcal{X}) \times |\mathcal{Y}| \mid y \in f(u) \text{ and } y \notin f(u') \text{ for any } u' \subsetneq u\}.$$

By Lemma 4.1, there is $s \in \mathscr{R}$ such that

$$s \cdot a = \sum_{(u,y)\in T} d_u \cdot a \cdot r_{\{y\}}$$

for all $a \in \mathscr{R}$. For any $u \in |X|$ and $a \in \|u\|_X$,

$$\begin{aligned} s \cdot a &= \sum_{(u',y)\in T} d_{u'} \cdot a \cdot r_{\{y\}} \\ &= \sum_{(u',y)\in T \text{ and } u'\subseteq u} r_{\{y\}} \\ &= \bigvee_{(u',y)\in T \text{ and } u'\subseteq u} r_{\{y\}} \\ &= r_{f(u)}. \end{aligned}$$

Hence, s realizes f. □

We define a category $\mathbf{AsmC}(\mathscr{R})$ as follows.

- Object are coherent assemblies.
- Morphisms from $(X, \mathcal{X}, \theta)$ to $(Y, \mathcal{Y}, \xi)$ are realizable functions from X to Y.

It follows from Lemma 6.3 and Lemma 6.4 that $\mathbf{AsmC}(\mathscr{R})$ is equivalent to a full subcategory of $\mathbf{Asm}(\mathscr{R})$ and is also equivalent to a full subcategory of **Coh**.

Proposition 6.1. *The category* $\mathbf{AsmC}(\mathscr{R})$ *is cartesian.*

Proof. An $\mathbf{AsmC}(\mathscr{R})$-object consisting of

- the terminal object $(\{\emptyset\}, \|\emptyset\| = \{\varnothing\})$ in $\mathbf{Asm}(\mathscr{R})$,
- the terminal object $\emptyset$ in **Coh**,
- $\mathrm{id}_{\{\emptyset\}}$

is a terminal object of $\mathbf{AsmC}(\mathscr{R})$. For $\mathbf{AsmC}(\mathscr{R})$-objects $(X, \mathcal{X}, \theta)$ and $(Y, \mathcal{Y}, \xi)$, we define $(X, \mathcal{X}, \theta) \times (Y, \mathcal{Y}, \xi)$ by

- $X \times Y$,
- $\mathcal{X} \times \mathcal{Y}$,
- $\chi \colon |X| \times |Y| \to \mathcal{C}(\mathcal{X} \times \mathcal{Y})$ given by

$$\chi(u, v) = \{(0, x) \mid x \in \theta(u)\} \cup \{(1, v) \mid y \in \xi(v)\}$$

is a product of $(X, \mathcal{X}, \theta)$ and $(Y, \mathcal{Y}, \xi)$. For $(u, v) \in |X| \times |Y|$, $r_{(u,v)} \in \|(u, v)\|$ given by

$$r_{(u,v)} = r_u \boxplus r_v$$

satisfy the conditions in Definition 6.1. For a finite $(u, v) \in |X| \times |Y|$, $\sigma_{(u,v)} \colon X \times Y \to \Sigma$ is realized by

$$\lambda x.\, d_u \cdot (\mathsf{fst} \cdot x) \cdot (d_v \cdot (\mathsf{snd} \cdot y)).$$

□

Proposition 6.2. *The category* $\mathbf{AsmC}(\mathscr{R})$ *is cartesian closed.*

Proof. For $\mathbf{AsmC}(\mathscr{R})$-objects $(X, \mathcal{X}, \theta)$ and $(Y, \mathcal{Y}, \xi)$, we define $(X, \mathcal{X}, \theta) \Rightarrow (Y, \mathcal{Y}, \xi)$ by

- $X \Rightarrow Y$,
- $\mathcal{X} \Rightarrow \mathcal{Y}$,
- $\chi\colon |X \Rightarrow Y| \to \mathcal{C}(\mathcal{X} \Rightarrow \mathcal{Y})$ given by

$$\chi(f) = \text{the trace of } \xi \circ f \circ \theta^{-1}.$$

By Lemma 6.3 and Lemma 6.4, the definition of χ makes sense. We check that this is a coherent assembly. Let $u_1, u_2, \ldots$ be an enumeration of elements in $\mathcal{C}_{\text{fin}}(\mathcal{X})$, and let $y_1, y_2, \ldots$ be an enumeration of elements in $|\mathcal{Y}|$. For a realizable $f\colon X \to Y$, we choose $r_f \in \mathscr{R}$ such that

$$r_f \cdot x = \sum_{n \in \mathbb{N}} r_{n,f} \cdot x$$

where

$$r_{n,f} = \begin{cases} \lambda x.\, d_{u_n} \cdot x \cdot r_{\{y_n\}}, & \text{if } (u_n, y_n) \in \chi(f), \\ \varnothing, & \text{otherwise.} \end{cases}$$

Like in Lemma 6.4, we can show that r_f is a realizer of f. It follows from the construction in the proof of Lemma 4.1, we can choose r_f so that r_f satisfies the requirements in Definition 6.1. For $(u, y) \in |\mathcal{X} \Rightarrow \mathcal{Y}|$, $d_{(u,y)}$ given by

$$\lambda x.\, d_{\{y\}} \cdot (x \cdot r_u)$$

realizes $\sigma_{(u,y)}\colon X \Rightarrow Y \to \Sigma$. It is straightforward to generalize this construction to arbitrary finite cliques of $\mathcal{X} \Rightarrow \mathcal{Y}$. □

Theorem 6.1. *Let* $\mathbf{A}$ *be the full cartesian closed subcategory of* $\mathbf{Asm}(\mathscr{R})$ *generated by* N. *Then* $\mathbf{A}$ *is equivalent to a full subcategory of* $\mathbf{Coh}$.

Proof. By Proposition 6.2 and Lemma 6.3 and Lemma 6.4, the projection functors $P_1\colon \mathbf{AsmC}(\mathscr{R}) \to \mathbf{Asm}(\mathscr{R})$ and $P_2\colon \mathbf{AsmC}(\mathscr{R}) \to \mathbf{Coh}$ are fully faithful cartesian closed functors. It is straightforward to check $\mathbf{A}$ is equivalent to a full cartesian closed subcategory of $\mathbf{AsmC}(\mathscr{R})$. □

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